Fermi National Accelerator Laboratory

FERMILAB-Conf.-90/240-A
November 1990

STOCHASTIC INFLATION LATTICE SIMULATIONS: ULTRA-LARGE SCALE STRUCTURE OF THE UNIVERSE

D.S. Salopek
NASA/ Fermilab Astrophysics Center
P.O. Box 500 MS-209
Batavia, Illinois, USA 60510

ABSTRACT

Non-Gaussian fluctuations for structure formation may arise in inflation from the nonlinear interaction of long wavelength gravitational and scalar fields. Long wavelength fields have spatial gradients $a^{-1}\nabla$ small compared to the Hubble radius, and they are described in terms of classical random fields that are fed by short wavelength quantum noise. Lattice Langevin calculations are given for a 'toy model' with a scalar field interacting with an exponential potential where one can obtain exact analytic solutions of the Fokker-Planck equation. For single scalar field models that are consistent with current microwave background fluctuations, the fluctuations are Gaussian. However, for scales much larger than our observable Universe, one expects large metric fluctuations that are non-Gaussian. This example illuminates non-Gaussian models involving multiple scalar fields which are consistent with current microwave background limits.

in Proceedings of *IUPAP Conference:*
Primordial Nucleosynthesis and Evolution of the Early Universe
Tokyo, Japan, September 4-8, 1990
eds. K. Sato and J. Yokoyama, Kluwer Academic Publishers.

1. INTRODUCTION

There are a growing number of cosmological observations[1-5] that are in apparent conflict with the simplest model of structure formation, the Cold Dark Matter model. One possibility is that the Gaussian initial conditions described by a Zeldovich scale-invariant spectrum that typically arise from inflation are incorrect. The view that will be adopted here is that the inflation model is basically correct although one should refine the calculations to incorporate nonlinearities among multiple scalar fields. As the first step, nonlinearities will be incorporated at wavelengths larger than the Hubble radius. The initial conditions for structure formation are generated stochastically as quantum modes expand beyond the Hubble radius. One can then investigate whether non-Gaussian fluctuations for structure formation can arise from inflation. I report the results of a collaboration[6,7] with J.R. Bond of the Canadian Institute for Theoretical Astrophysics, Toronto, Canada.

There were two clues that indicated that the nonlinear evolution of long wavelength fields was tractable. (1) For a single scalar field, Bardeen, Steinhardt and Turner[8] demonstrated using linear perturbation theory that there was a remarkable constant of integration ζ when the physical wavelength of a comoving mode exceeded the Hubble radius. (2) When two spatial points are separated by more than the Hubble radius, they are no longer in causal contact; they essentially evolve as independent Universes. Hence, linear perturbation theory is not essential. In a significant improvement over homogeneous mini-superspace models, one can in fact generalize ζ to nonlinear multiple fields.[6]

One begins by decomposing all fields into long wavelength (denoted by a bar) and short wavelength components (denoted by δ),

$$\phi_j(T,x) = \bar{\phi}_j(T,x) + \delta\phi_j(T,x), \quad g_{\mu\nu}(T,x) = \bar{g}_{\mu\nu}(T,x) + \delta g_{\mu\nu}(T,x), \tag{1.1}$$

where the Hubble radius, H^{-1}, marks the boundary between long and short. The metric will be assumed to have the diagonal form,

$$ds^2 = -N^2(T,x)dT^2 + e^{2\alpha(T,x)}\left((dx^1)^2 + (dx^2)^2 + (dx^3)^2\right),$$

described by a lapse function, $N(T,x)$, and an inhomogeneous scale factor, $e^{\alpha(T,x)}$. Hence, the dynamic effects of gravitational radiation are neglected which is typically an excellent approximation. The long wavelength scalar and gravitational field equations will be solved nonlinearly (Sec. 2) whereas the short wavelength modes will still (unfortunately) be treated in linear perturbation theory (Sec. 3). As the comoving short wavelength modes expand beyond the Hubble radius, one assumes that they become classical and they add a stochastic kick[9-12] to the long wavelength background.

2. EVOLUTION OF LONG WAVELENGTH FIELDS

The long wavelength ADM equations that will be required in this report are,

$$\bar{H}^2 = \frac{8\pi}{3m_P^2}\left(\frac{1}{2}\sum_l \bar{\Pi}^{\phi_l 2} + V(\bar{\phi}_j)\right) \tag{2.1a}$$

$$\bar{H}_{,i} = -\frac{4\pi}{m_P^2}\sum_l \bar{\Pi}^{\phi_l}\bar{\phi}_{l,i} \tag{2.1b}$$

$$\Delta\bar{\phi}_j = \bar{\Pi}^{\phi_j}\bar{N}\Delta T + \Delta S_{\phi_j}. \tag{2.1c}$$

$\bar{H}$ is the Hubble parameter, whereas the $\bar{\Pi}^{\phi_j}$ are the scalar field momenta. Eqs. (2.1) follow from the standard scalar field equations and Einstein's equations with all second order spatial gradients neglected. First order spatial gradients are retained otherwise one is describing homogeneous mini-superspace which is too limited for the applications considered. The first two equations, the energy constraint and the momentum constraint, do not explicitly contain noise terms, whereas the third one, the evolution equations for $\bar{\phi}_j$ contain a contributions from short wavelength quantum noise, ΔS_{ϕ_j}, that have crossed the Hubble radius. The remaining evolution equations for $\bar{\Pi}^{\phi_j}$ and $\bar{H}$ also contain noise terms but they are not required if one neglects decaying modes. The lapse function will be specified when one chooses the time parameter (Sec. 3).

The crucial step in solving eqs.(2.1) is to integrate the momentum constraint. The Hubble parameter is a function of the scalar fields and possibly the time parameter, T:

$$\bar{H} \equiv \bar{H}(\bar{\phi}_k(x), T), \quad \text{where} \quad \bar{\Pi}^{\phi_k} = -\frac{m_{\mathcal{P}}^2}{4\pi}\frac{\partial \bar{H}}{\partial \bar{\phi}_j}(\bar{\phi}(x), T). \tag{2.2}$$

(If noise term is not important, one can actually show that the Hubble parameter does not depend explicitly on time,[6] $\bar{H} \equiv \bar{H}(\bar{\phi}_k)$.) For example, the spatial derivative of the Hubble parameter in (2.1b) may be expanded leading to,

$$\frac{\partial \bar{H}}{\partial \bar{\phi}_j}\bar{\phi}_{j,i} = -\frac{4\pi}{m_{\mathcal{P}}^2}\bar{\Pi}^{\phi_j}\bar{\phi}_{j,i},$$

which may satisfied if one identifies the scalar field momenta with the partial derivative of $\bar{H}$ as in (2.2). Substituting the momenta into the energy constraint leads to the separated Hamilton-Jacobi equation (SHJE),

$$\bar{H}^2 = \frac{m_{\mathcal{P}}^2}{12\pi}\sum_l\left(\frac{\partial \bar{H}}{\partial \bar{\phi}_l}\right)^2 + \frac{8\pi}{3m_{\mathcal{P}}^2}V(\bar{\phi}_j), \tag{2.3}$$

a partial differential equation which does not depend explicitly on the time coordinate nor on the spatial variables. In this sense, it is a completely covariant equation. The momentum constraint essentially patches together the various spatial points to make one Universe.

The separated Hamilton-Jacobi equation is self-contained. Its solution is not unique, but in many cases of physical interest there is typically an attractor solution to which almost all solutions approach. For example, for a single scalar field interacting through an exponential potential,[13]

$$V(\phi) = V(0)\exp\left[-\sqrt{\frac{16\pi}{p}}\frac{\phi}{m_{\mathcal{P}}}\right], \tag{2.4}$$

the attractor solution[6] of (2.3) is

$$\bar{H}_{att}(\bar{\phi}) \equiv \bar{H}(0)\exp\left(-\sqrt{\frac{4\pi}{p}}\frac{\bar{\phi}}{m_{\mathcal{P}}}\right), \quad \bar{H}(0) = \left[\frac{8\pi V(0)}{3m_{\mathcal{P}}^2(1-\frac{1}{3p})}\right]^{1/2}. \tag{2.5}$$

The parameter p describes the flatness of the potential; inflation occurs if $p > 1$. Of course, other solutions exist[6] but they describe the decaying mode that always appears in cosmological models. If one simply neglects the decaying mode, then the Hubble parameter of stochastic inflation may be identified with $\bar{H}_{att}(\bar{\phi}_k)$ and one may safely ignore the explicit time dependence appearing in (2.2). With this additional assumption, one need not consider the remaining evolution equations that were dropped in (2.1).

In fact, one of the big advantages of applying Hamilton-Jacobi theory to stochastic inflation is that growing and decaying modes are cleanly separated even when the slow-roll approximation is not valid. Another advantage is that gauge ambiguities are not as problematic as in linear perturbation theory because one does not introduce a fictitious homogeneous background. Since the constraints have been eliminated, only the evolution equations for the scalar fields, eq.(2.1c), remain to be solved. Given the attractor solution of the SHJE, $\bar{\Pi}^{\phi_k}$ is assumed to be a known function of $\bar{\phi}_k$ through (2.2). When the time parameter is specified in Sec. 3, the lapse function will also be a known function of the scalar fields.

The initial conditions for the long wavelength problem are generated by short wavelength quantum fluctuations whose wavelength exceeds the Hubble radius. By assuming that they become classical when they cross the horizon,[9,10] one circumvents the problem of quantization of the long wavelength gravitational field which appears to be inconsistent beyond the semi-classical approximation.[6,7] However, using an exact solution of the long wavelength Wheeler-DeWitt equation for single scalar field with an exponential potential (2.4), one can nonetheless estimate that quantum gravity corrections are of the order of

$$\text{Quantum Corrections} \sim 4\pi \frac{H^2}{m_{\mathcal{P}}^2}, \tag{2.6}$$

where H is the value of the Hubble parameter when the comoving scale of interest (typically $\approx 3000h^{-1}$Mpc) crossed the horizon during inflation. For single scalar field models that are consistent with microwave background limits, $\bar{H} \approx 10^{-5} m_{\mathcal{P}}$, the correction is approximately one in a billion. For many models, it is then an excellent approximation to treat the long wavelength fields classically, although a probabilistic description is essential.

3. QUANTUM NOISE FROM SHORT WAVELENGTH FIELDS

Using the stochastic long wavelength formalism, one may answer questions which could not be adequately addressed in homogeneous mini-superspace quantum cosmology: What is the time parameter? What is the initial choice of the probability distribution?

3.1 THE CHOICE OF TIME PARAMETER

In order to describe the evolution of short wavelength quantum fluctuations on an inhomogeneous long wavelength background, it proves convenient to choose conformal time τ as the time parameter so that the long wavelength metric has the form

$$ds^2 = e^{2\bar{\alpha}(\tau,x^j)}(-d\tau^2 + dx^2 + dy^2 + dz^2). \tag{3.1}$$

If the fields evolve, slowly then conformal time may shown to be given by

$$\tau \sim -\frac{1}{\bar{H}e^{\bar{\alpha}}}. \tag{3.2}$$

In practice, one employs

$$T = \ln(\bar{H} e^{\bar{\alpha}}) \tag{3.3}$$

as the time parameter because it is easier to apply in numerical analyses.

One can show that in linear perturbation theory about a long wavelength background, a Fourier mode solution for the scalar field is an excellent approximation,

$$\delta\phi_j(\tau, x) = e^{-\bar{\alpha}(\tau,x)} e^{i\mathbf{k}\cdot\mathbf{x}} e^{-ik\tau}/\sqrt{2k}, \tag{3.4}$$

when the physical wavelength $e^{\bar{\alpha}} k^{-1}$ is much shorter than the Hubble radius, H^{-1}. Eq.(3.4) corresponds to the positive energy solution that describes the quantum mechanical ground state. The effects of long wavelength inhomogeneities are contained in the spatially dependent background scale factor, $e^{-\bar{\alpha}(\tau,x)}$. At very short wavelengths, metric fluctuations beyond the long wavelength background are not important at least in linear perturbation theory. The amplitude is normalized in analogy to the analysis of linear quantum fluctuations on a homogeneous time dependent background[14] where one employs the equal time quantum commutator relations,

$$[\phi_j(\tau, x), \Pi^{\phi_k}(\tau, x')] = i\, e^{-2\bar{\alpha}}\, \delta^3(x - x')\, \delta_j^k.$$

When the physical wavelength of a mode approaches the Hubble radius the approximate solution (3.4) breaks down, but this is precisely when long wavelength evolution becomes important. At the start of the timestep, ΔT, one adds to the background a noise impulse, ΔS_{ϕ_j}, which consists of all those Fourier modes that will have crossed the Hubble radius during the time step,

$$\bar{\phi}_j(T, x) \;\rightarrow\; \bar{\phi}_j(T, x) + \Delta S_{\phi_j}, \quad \text{where} \quad \Delta S_{\phi_j} = \bar{H}\left[\bar{\phi}(T, x)\right] \sum_{e^T \le |\mathbf{k}| < e^{T+\Delta T}} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{2k^3}} a_j(\mathbf{k}), \tag{3.5}$$

In (3.5), I have used the notation,

$$\sum_{\mathbf{k}} \equiv \int \frac{d^3\mathbf{k}}{(2\pi)^3},$$

and I have applied the horizon crossing expression $e^{-\bar{\alpha}(T,x)} = \bar{H}(\bar{\phi}_j(T, x))/k$. Here, $a_j(\mathbf{k})$ is a classical complex Gaussian random field,

$$a_j^*(\mathbf{k}) = a_j(-\mathbf{k}) \quad \text{and} < a_j(\mathbf{k}) a_{j'}(\mathbf{k}') >= (2\pi)^3 \delta^3(\mathbf{k} + \mathbf{k}') \delta_{jj'}, \tag{3.6}$$

which imitates quantum fluctuations which are Gaussian in linear perturbation theory. The noise term (3.5) differs from the usual perturbation calculation on a homogeneous background in that it is modulated by the local value of the background Hubble parameter $\bar{H}\left[\bar{\phi}(T, x)\right]$.

The choice of time parameter (3.3) leads to the following definition of the lapse function,

$$\bar{N}^{-1} = \dot{T}/\bar{N} = \dot{\bar{\alpha}}/\bar{N} + \frac{1}{\bar{H}} \dot{\bar{H}}/\bar{N} = \bar{H} - \frac{m_P^2}{4\pi} \frac{1}{\bar{H}} \sum_m \left(\frac{\partial \bar{H}}{\partial \bar{\phi}_m}\right)^2, \tag{3.7}$$

The formulation of the stochastic problem is now complete. One solves (2.1c) numerically on a lattice. $\bar{\Pi}^{\phi_i}$ and $\bar{N}$ are expressed in terms of the attractor solution through (2.2) and (3.7). The noise term is given in (3.5).

4. STOCHASTIC INFLATION LATTICE SIMULATIONS

4.1 INITIAL CONDITIONS AND THE FOKKER-PLANCK EQUATION

A lattice simulation of inflation begins homogeneously at $T = T_0$ with $\bar{\phi}_j = \bar{\phi}_{j0}$; inhomogeneities are produced only subsequently by quantum noise. If $P(\bar{\phi}_j|T;\bar{\phi}_{j0},T_0)$ denotes the scalar field distribution on a uniform T surface, then the initial probability distribution for the scalar field is a δ function:

$$P(\bar{\phi}_j|T;\bar{\phi}_{j0},T_0) = \delta^n(\bar{\phi}_j - \bar{\phi}_{j0}). \tag{4.1}$$

The probability function P is an example of a limited statistic that gives a partial understanding of a complicated lattice simulation. From now on, only long wavelength fields will be considered and thus the bar notation will be dropped.

For a single scalar field with an exponential potential (2.4), the Fokker-Planck equation that describes the evolution of the probability function with the full metric back reaction is then

$$\begin{aligned}\frac{\partial P}{\partial T} = &- \frac{m_{\mathcal{P}}}{\sqrt{4\pi p}}\frac{1}{1-1/p}\frac{\partial P}{\partial \phi} + \frac{H^2(0)}{8\pi^2}\frac{\partial^2}{\partial\phi^2}\left[\exp\left(-\sqrt{\frac{16\pi}{p}}\frac{\phi}{m_{\mathcal{P}}}\right)P\right] \\ &+ \delta(T-T_0)\delta(\phi-\phi_0).\end{aligned} \tag{4.2}$$

The first term on the right hand side describes the classical drift of the scalar field down the potential, whereas the second, $(\partial^2/\partial\phi^2)(H^2_{att}(\phi)P)/(8\pi^2)$, is the result of quantum noise that causes the diffusion of the probability function. The third term incorporates the initial conditions (4.1). There are factor ordering problems and other corrections associated with this equation,[7] but I will ignore them here. The solution of (4.2) will be referred to as the Green's function:[7]

$$P(\phi|T;T_0,\phi_0) = \sqrt{\frac{16\pi}{pm^2_{\mathcal{P}}}}\, y^{-1}e^{-(1+z^2)/y}\, z^2\, I_0(2z/y). \tag{4.3a}$$

where I_0 is the modified Bessel function of order zero, $I_0(x) = J_0(ix)$. The functions $z(\phi,T)$ and $y(T)$ are given by

$$z(\phi,T) = \exp\left(\sqrt{\frac{4\pi}{p}}\frac{\phi-\phi_0}{m_{\mathcal{P}}} - \frac{T-T_0}{p-1}\right), \tag{4.3b}$$

$$y(T) = \frac{1}{\pi}(1-1/p)\frac{H^2(\phi_0)}{m^2_{\mathcal{P}}}\left[1 - e^{-2(T-T_0)/(p-1)}\right]. \tag{4.3c}$$

(The coefficient $H(\phi_0)$ should not be confused with the $H(0)$ of (2.5), which it equals only for $\phi_0 = 0$). At late times, $T - T_0 \to \infty$, quantum diffusion is no longer important because the stochastic force is proportional to the Hubble parameter which decreases in time, and the probability distribution on the lattice evolves as a wave of fixed shape,

$$\lim_{T-T_0\to\infty} P(\phi|T;T_0,\phi_0) = f_{\phi_0}\big(\phi-\phi_0-\frac{m_{\mathcal{P}}}{\sqrt{4\pi p}}\frac{1}{(1-1/p)}(T-T_0)\big).$$

The Fokker-Planck equation for an exponential potential was first analyzed by Ortolan, Matarrese and Lucchin[15,16] who applied both analytic approximations was well as numerical methods. Eq. (4.3) is the *exact* Green's function solution of the improved Fokker-Planck equation (4.2).

4.2 NUMERICAL SIMULATIONS

The late time results, $T - T_0 \to \infty$ of two 64^3 lattice simulations for a single scalar field interacting through an exponential potential with p=5 are shown in Figs. 1-3. In Fig. 1a, I show contour plots on a surface of constant T corresponding to -2, -1, 0, 1, 2 σ scalar field fluctuations from the mean for a lattice simulation that began homogeneously with $H(\phi_0) = 10^{-5}m_{\mathcal{P}}$, consistent with microwave background anisotropy limits. In Fig. 2a, only -2σ fluctuations are shown for the benefit of the reader. If one included the effects of beam smearing, these figures would correspond to microwave background maps at angular scales greater than 2^0. One may actually perform a hypersurface transformation to show that $\Delta\phi$ on a uniform T slice is actually proportional to the microwave background anisotropy,[7]

$$\Delta T_{cmb}/T_{cmb} = \sqrt{4\pi p}\,(1-1/p)\,\Big(\Delta\phi(T)/m_{\mathcal{P}}\Big)/5\,. \tag{4.4}$$

In Fig. 3a, for a constant value of T, I have plotted the distribution on the lattice of the scalar field variable

$$\chi = (\phi - \phi_{cl})/H(\phi(0)) \tag{4.5}$$

where φ_{cl} is the classical attractor trajectory,

$$\phi_{cl}(T) = \phi_0 + \frac{m_{\mathcal{P}}}{\sqrt{4\pi p}}\frac{1}{1-1/p}(T-T_0)\,.$$

Agreement with the exact Green's function solution (4.3) (broken curve) is excellent, and the distribution is Gaussian to an excellent approximation.

Non-Gaussian fluctuations arise when the initial value of the Hubble parameter is comparable to the Planck scale. In Fig. 3b, I show for $H(\phi_0) = 1.0m_{\mathcal{P}}$ that the histogram of scalar field values from a lattice calculation agrees well with the exact solution. Once again, contour plots for -2, -1, 0, 1, 2 σ fluctuations from the mean are shown in Fig. 1b whereas only -2σ contours are given in Fig. 2b. The non-Gaussian contours look remarkably different from the Gaussian case, essentially because of the extended high energy density tail that appears in Fig. 1b. These points have been able to diffuse to relatively high energy densities because the stochastic force is proportional to the Hubble parameter. Figs. 1 and 2 are plotted using comoving spatial variables, x, whereas if one used physical space, then the high energy density regions would actually dominate the volume because they have almost eternally inflated.[12] In fact, a numerical instability occurs when $H(\phi_0) > m_{\mathcal{P}}$

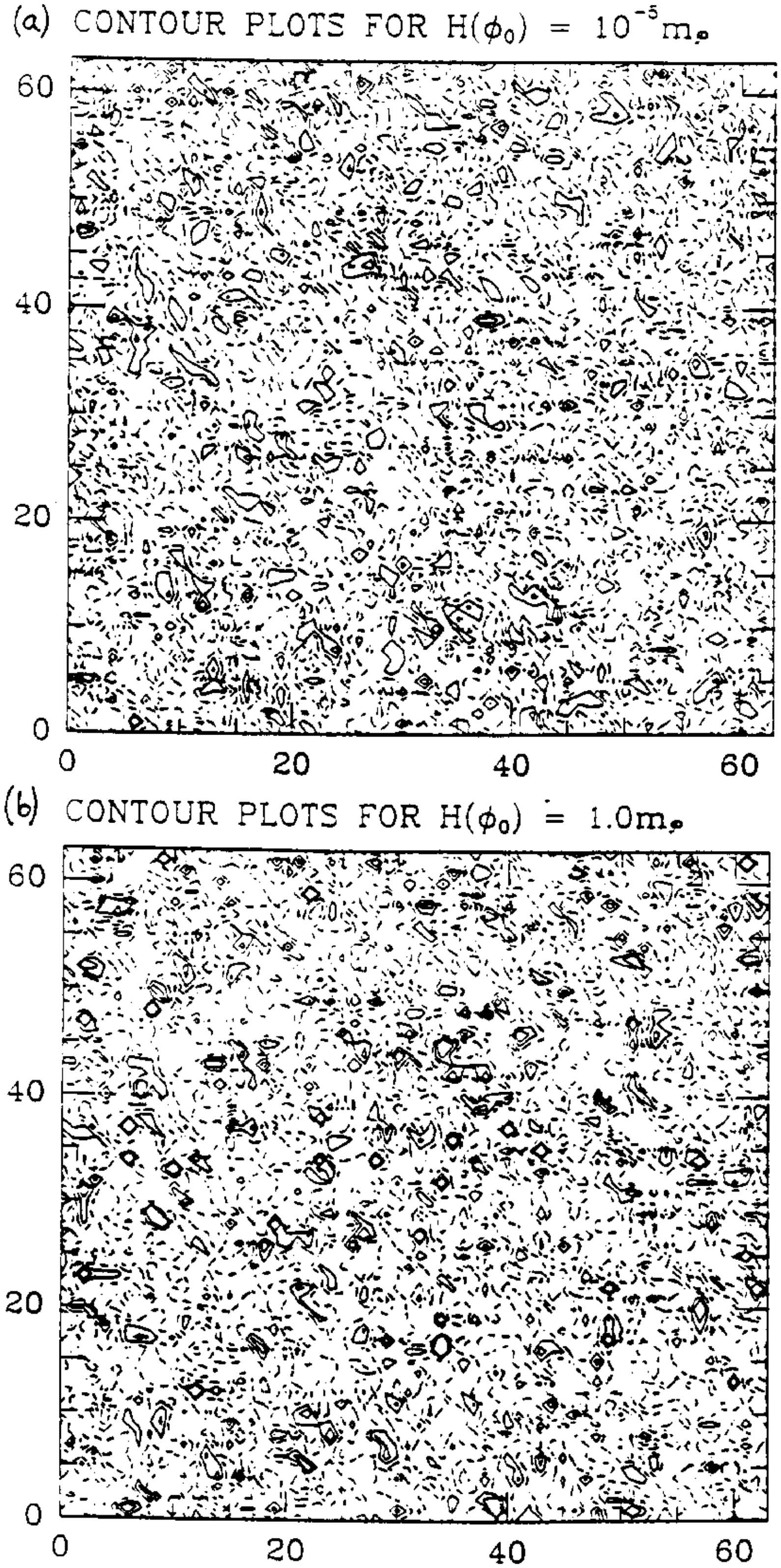


Fig. 1 Contour maps of scalar field fluctuations for a two dimensional slice of a 64^3 lattice simulation for an exponential potential stochastic inflation model with $p = 5$. The initial configurations were homogeneous, with $H(\phi_0)/m_P = 10^{-6}$ for (a) and $H(\phi_0) = 1.0 m_P$ for (b). The solid contours correspond to -2σ and -1σ deviations from the scalar field mean, (*i.e.*, high energy density regions) and the broken contours correspond to 0, 1, 2σ fluctuations. The mean has been subtracted out. The initial condition for (a) was chosen to yield scalar field fluctuations that lead to structure compatible with current microwave background anisotropy limits; the fluctuations are Gaussian-distributed to high accuracy. (b) is one of the simplest models where non-Gaussian statistics can arise in cosmology. The map is in initial comoving position rather than final physical position and has a uniform value of $H(\phi)e^{\alpha} = e^{T}$, where T is the time at which the slice is viewed. Because fluctuations are much larger than allowed by present microwave background limits, the size of the lattice is much larger than our present horizon size, and as a result this map has no observable consequences.

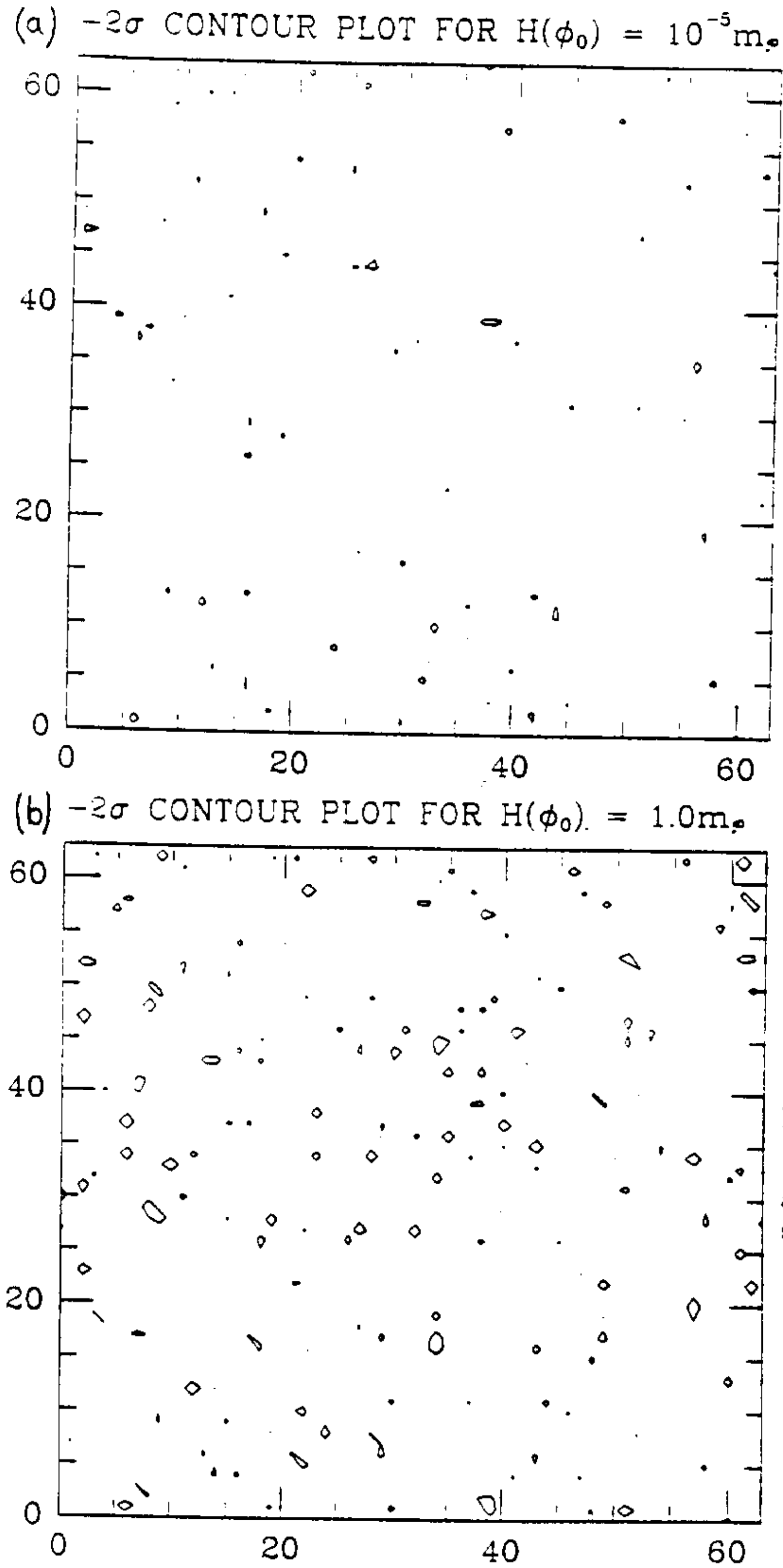
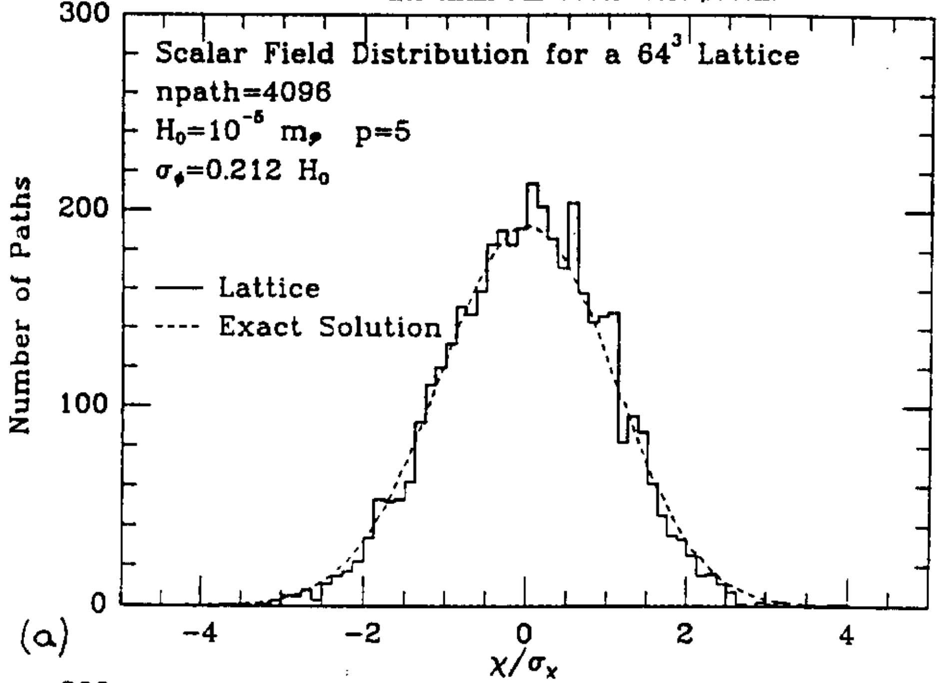
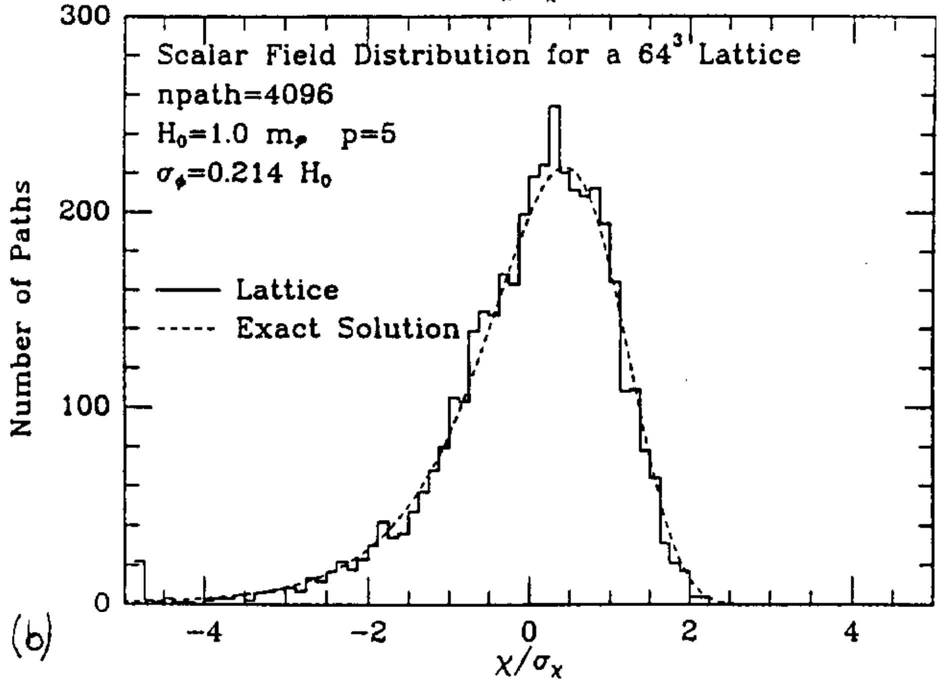


Fig. 2 This figure shows only the -2σ contours of the scalar field distribution for the same cases as in Fig.1. In (a), there are 91 points with $\phi < -2\sigma$, which is approximately what is predicted from Gaussian statistics (94). For (b), because of the long tail in Fig. 3, the high energy density areas with negative values of the scalar field are more numerous than in the Gaussian example of (a): there are 192 points with $\phi \leq -2\sigma$. The high energy density regions are more strongly clustered at a shorter distance scale. In alternative models which could describe our observable Universe, a significant non-Gaussian tail would have an important impact on structure formation.

Fig. 3 As a check of the numerical method, the final scalar field distributions from the lattice simulations of Fig.1a,b (solid histogram) are shown to agree with the exact solutions of eq.(4.3) (dashed curves). For (a), with $H(\phi_0) = 10^{-6} m_{\mathcal{P}}$, this is a Gaussian distribution. In (b), of the initially $64^3 = 4086$ points, 22 paths wandered beyond χ_{cut}, eq.(4.6), and were discarded, as described in Sec. 4.2. Simulations with even larger $H(\phi_0)$ have a much larger loss of trajectories: *e.g.*, with $H(\phi_0) = 3m_{\mathcal{P}}$, approximately 10% of the points are discarded. In single scalar field models, significant non-Gaussian distributions are generated only if the scalar field begins with a Hubble parameter $H(\phi_0) \approx m_{\mathcal{P}}$, corresponding to a patch of the Universe $\approx 10^{20}$ times larger than our observable patch.



(a)



(b)

because some lattice points may actually diffuse to $\phi = -\infty$ in a finite period of time.[7] Here, I have simply removed from the lattice all those points that decrease below χ_{cut},

$$\chi_{cut} \equiv -5\sqrt{\frac{p-1}{8\pi^2}}. \tag{4.6}$$

This simulation with its large initial value of the Hubble parameter cannot describe our observable Universe otherwise microwave background limits which are determined by (4.4) would be violated. In fact, the lattice size in Figs. 1b, 2b is more than 10^{20} times larger than that in Figs. 1a, 2a because as the scalar field rolls down from $1.0m_{\mathcal{P}}$ to $10^{-5}m_{\mathcal{P}}$, the Universe expands by a factor of

$$\text{Relative Size} = (10^5)^{p-1} \sim 10^{20}, \quad \text{for} \quad p = 5.$$

The relative size depends sensitively on the free parameter p, eq.(2.4), which also controls the slope of the primordial fluctuation spectrum in the Newtonian potential,[7]

$$\mathcal{P}_{\Phi_H} \propto k^{-2/(p-1)}.$$

$p = \infty$ is the scale-invariant Zeldovich spectrum. If one normalizes the Cold Dark Matter fluctuation spectrum at galaxy scales, then p cannot be much smaller than 5 otherwise large angle microwave background limits would be violated in Figs. 1a, 2a.[7]

5. DISCUSSION AND CONCLUSIONS

Theoretical large scale structure models have not kept pace with the growing number of cosmological observations. It is a challenge to the theorist to propose alternative scenarios which are consistent with the observations. Here, I have summarized the first steps towards producing models which produce calculable non-Gaussian fluctuations from inflation. Non-Gaussian fluctuations can arise from nonlinear long wavelength evolution whose signature may perhaps be observed in the near future from microwave background fluctuations.

Typically for single scalar field models, the nonlinear evolution of long wavelength fields does not produce significant non-Gaussian fluctuations in our observable Universe because microwave background anisotropy limits force the initial value of the Hubble parameter to be quite small. However, for scales much larger than our observable Universe, non-Gaussian fluctuations do arise from lattice points that almost eternally inflate. The stochastic formalism has been sufficiently developed that one can now consider multiple scalar field models that produce non-Gaussian fluctuations in our observable Universe.[17]

Exact solutions of Starobinski's[10] Fokker-Planck equation have been given for a $\lambda\phi^4/4$ potential by Yi, Vishniac and Mineshige (YVM).[18] In their case non-Gaussian fluctuations can be significant at moderate deviations from the mean $\sim 6\sigma$ even when $H(\phi_0) \sim 10^{-3}m_{\mathcal{P}}$. For the case of an exponential potential, a careful numerical analysis of eq.(4.3) shows that non-Gaussian fluctuations are significant at 6σ for $H(\phi_0) > 0.02m_{\mathcal{P}}$. (By significant, I mean more than a 100% change in the probability distribution from a Gaussian one.) This mild discrepancy may signal that a $\lambda\phi^4/4$ is intrinsically different than a exponential potential. It would be interesting to see whether one could generalize their method of solution to the improved Fokker-Planck equation which includes the full nonlinear metric back-reaction.[7]

In any case, the YVM effect is typically small because microwave background limits require a much smaller value of the Hubble parameter, $H(\phi_0) \lesssim 10^{-5} m_{\mathcal{P}}$. Furthermore, 2-3$\sigma$ fluctuations are typically of greater interest for the formation of galaxies.[19]

In the inflation model, quantum fluctuations in the scalar field are converted into metric fluctuations. Hence, one ultimately, requires a quantum theory of the gravitational field if one wishes to extend the formalism describing inflation. However, attempts to model spatial variations have employed at most linear perturbations on a homogeneous background.[20,14] In addition, quantization of the gravitational field using the Wheeler-DeWitt equation has encountered numerous difficulties including interpretation of negative probabilities, choice of time parameter and initial probability distribution, operator ordering problems, over-simplified models, *etc.* (consult ref. 21 for a recent review). Unfortunately, an attempt to a construct a quantum theory of long wavelength fields appears to be inconsistent with the momentum constraint beyond the semi-classical approximation.[6] In any case, one can give arguments that suggest that quantum corrections are typically small at long wavelengths. For practical purposes, one can simply assume that short wavelength quantum fluctuations become classical when the wavelength exceeds the Hubble radius. In this context, the Fokker-Planck equation and the associated Langevin equation have proven to be more useful than the Wheeler-DeWitt equation because they are solvable and because they admit a simple interpretation in terms of initial conditions for structure formation.

I would like to thank J.R. Bond for a fruitful collaboration on these stochastic inflation lattice topics. This work was supported by the U.S. Department of Energy and NASA at Fermilab (Grant No. NAGW-1340).

6. REFERENCES

[1] Bahcall, N. and Soneira, R., Astrophys. J.**270**, 70 (1983).

[2] De Lapparent, V., Geller, M.J. and Huchra, J.P., Ap. J. **302**, L1 (1986).

[3] Dressler, A., Faber, S.M., Burstein, D., Davies, R.L., Lynden-Bell, D., Terlevich, R.J. and Wegner, G., Astrophys. J. Lett. **313**, L37 (1986).

[4] Maddox, S.J., Efstathiou, G., Sutherland, W.J. and Loveday, J., Mon. Not. R. Astr. Soc., **242**, 43P (1990).

[5] Broadhurst, T.J., Ellis, R.S., Koo, D.C. and Szalay, A.S., Nature **343**, 726 (1990).

[6] Salopek, D.S. and Bond, J.R., *Nonlinear Evolution of Long Wavelength Metric Fluctuations in Inflation Models*, Phys. Rev. **D42** (in press, 1990).

[7] Salopek, D.S. and Bond, J.R., *Stochastic Inflation and Nonlinear Gravity*, Phys. Rev. D (in press, 1990).

[8] Bardeen, J.M., Steinhardt, P.J. and Turner, M.S., Phys. Rev. **D28**, 670 (1983).

[9] Vilenkin, A., Phys. Rev. **D27**, 2848 (1983).

[10] Starobinski, A.A., in *Current Topics in Field Theory, Quantum Gravity, and Strings*, Proc. Meudon and Paris VI, ed. H.T. de Vega and N. Sanchez **246** 107 (Springer-Verlag, 1986)

[11] Bardeen, J.M. and Bublik, G.J., Class. Quant. Grav. **5**, L113 (1988).

[12] Goncharov, A.S., Linde, A.D. and Mukhanov, V.F., Intern. Jour. Mod. Phys. **A2**, 561 (1987).

[13] Lucchin, F. and Matarrese, S., Phys. Rev. **D32**, 1316 (1985).
[14] Salopek, D.S, Bond, J.R. and Bardeen, J.M., Phys. Rev. **D40**, 1753 (1989).
[15] Ortolan, A., Lucchin, F. and Matarrese, S., Phys. Rev. **D38**, 465 (1988).
[16] Matarrese, S., Ortolan, A. and Lucchin, F. Phys. Rev. **D40**, 290 (1989).
[17] Salopek, D.S., in progress, (1991).
[18] Yi, I., Vishniac, E.T. and Mineshige, S., University of Texas Preprint (1990).
[19] Bardeen, J.M., Bond, J.R., Kaiser, N. and Szalay, A.S., Ap.J. **304**, 15 (1986).
[20] Halliwell, J.J. and Hawking, S., Phys. Rev. **D31**, 1777 (1985).
[21] Halliwell, J.J., Int. J. Mod. Phys. **A5**, 2473 (1990).